Calculations of The Shape Trajectories of Vehicles and The Ackermann Principle of Steering

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Dedicated to the 195th anniversary of the great invention

Abstract

This paper investigates the output of invariant equations connection between the control of a vehicle and the trajectory of its movement. A control function is the angle Ackerman, defined as the difference between the angles of rotation of the front wheels of the car. The relations are illustrated by calculating the radius of the curve of the trajectory of the moving car and the required space for the car to move safely.

Key words: dynamical systems, nonholonomic systems, Ackerman principle of steering, the Ackerman angle, the trajectory of a vehicle, secure position corridor, perfect parking, mobile robots.

1 Introduction

Technical possibilities of control of dynamic systems based on the use of modern technologies have caused the recent emergence of a large number of works on the theory of control of the vehicle. The tasks of navigation [1,2], the control of the trajectory of the movement of [3-7], autonomous driving and maneuvering with small bending radius of a turn [8,9] are aimed at improving the safety and to increase the comfort, stimulated the emergence in recent years of new mathematical models. Most, but surprisingly not all, studies such as these to one degree or another are based on The Ackermann Principle of Steering - a great invention made 195 years ago.

2 Historical information

Rudolph Ackermann (born on 20th of April 1764 in Stolberg, Electorate of Saxony – died on 30th of March 1834 in London) was an Anglo-German bookseller, an inventor, a lithographer, a publisher and a businessman. He was born at Stolberg, in Saxony, where he attended the Latin school. His wish to study at the university was made impossible by the lack of financial means, and he therefore became a saddler like his father. In 1801 he patented a method for rendering paper and waterproof cloth and erected a factory in Chelsea to make it. He was one of the first to illuminate his own premises with gas. Indeed the introduction of lighting by gas owed much to him. After the Battle of Leipzig, Ackermann collected nearly a quarter of a million pounds sterling for the German casualties. He later began to manufacture colors and thick carton paper for landscape and miniature painters. He published many illustrated volumes of topography and travel, including The Microcosm of London (3 volumes, 1808–111), Westminster Abbey (2 volumes, 1812), The Rhine (1820) and The World in Miniature (43 volumes, 1821–6). He also patented the Ackermann steering geometry [10].
http://en.wikipedia.org/wiki/Rudolph_Ackermann

Rudolph Ackermann, portrait by François Mouchet between 1810 and 1814. (National Portrait Gallery, London)

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Ackermann’s Room by C.A. Pugin ca. 1809
Source: V&A (Museum No. E.3027-1903)

Son of a Bavarian coach builder, had spent a number of years designing coaches for English gentlemen in London, where he made his home. One of his more notable commissions was for the design of Admiral Nelson's funeral car in 1805. The Ackermann steering linkage was not actually Ackermann's invention, although he took out the British patent [10] (195 years ago) in his name and promoted the introduction of the running gear of which the linkage was a part (Fig. 1). The actual inventor was Ackermann's friend George Lankensperger of Munich, coach maker to the King of Bavaria [11]. . The advantage of being able to turn a carriage around in a limited area without danger of oversetting was immediately obvious, and while there was considerable opposition by English coach makers to an innovation for which a premium had to be paid, the invention soon "made its way from its own intrinsic merit," as Ackermann predicted it would. "Safety, durability, efficiency and comfort characterize the useful invention"

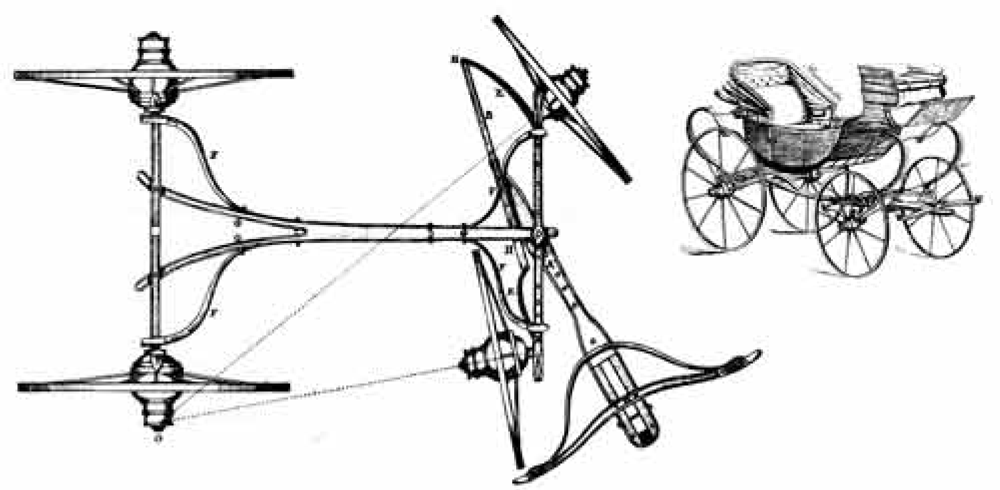

FIGURE 1.—Ackermann steering linkage of 1818, currently used in automobiles.

3 The law of steering

The traditional description principle of the Ackermann steering **c**onsider a front-wheel-steering 4W S vehicle that is turning to the left, as shown in Figure 2 [12]

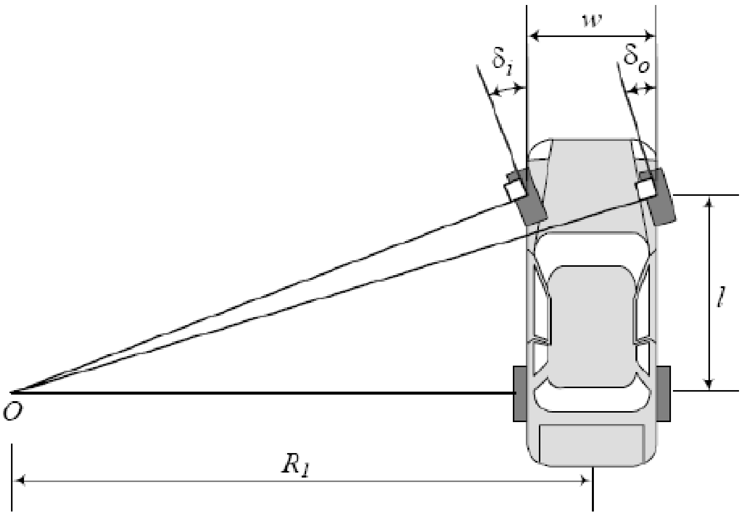

FIGURE 2. A front-wheel-steering vehicle and the Ackerman condition.

When the vehicle is moving very slowly, there is a kinematic condition between the inner and outer wheels that allows them to turn slip-free. The condition is called the Ackerman condition and is expressed by

$$\cot\delta_o - \cot\delta_i = \frac{w}{l}$$

Where δ_i the steer angle of the inner is *wheel*, and δ_o is the steer angle of the *outer wheel*. The inner and outer wheels are based on the turning center O.

Figure 3 illustrates a vehicle turning left. So, the turning center O is on the left, and the inner wheels are the left wheels that are closer to the center of rotation.

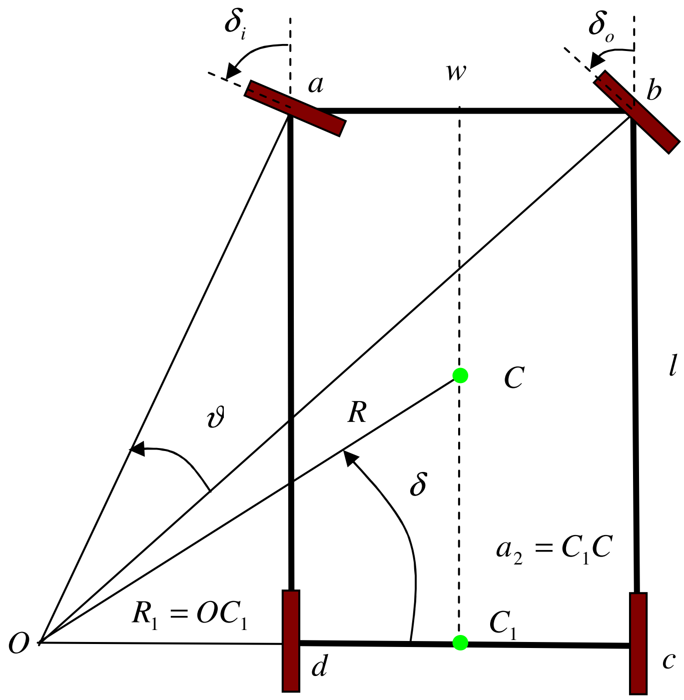

FIGURE 3. A front-wheel-steering vehicle and steer angles of the inner outer wheels and angle Ackerman

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The distance between the steer axes of the steerable wheels is called the *track* and is shown by w. The distance between the front and real axles is called the *wheelbase* and is shown by l. Track w and wheelbase l are considered as kinematic width and length of the vehicle.
The mass center of a steered vehicle will turn on a circle with radius R

$$R = \sqrt{a_2^2 + l^2 \cot^2 \delta},$$

where δ is found using the inner and outer steer angles.

$$\cot \delta = \frac{\cot \delta_o + \cot \delta_i}{2}$$

The following description works when one is solving a problem of building a vehicle, but it doesn't work if one is solving a problem of navigation, control of the vehicle or the maneuvering. For solving those problems one has to take into account another control parameter, such as the difference of the turning angles of the right and the left front wheels

$$\vartheta = \delta_l - \delta_r,$$

where δ_l is the steer angle of the *left wheel* , and δ_r is the steer angle of the *right wheel*. When the car takes a turn to the left, the following occurs:

$$\delta_l = \delta_i, \quad \delta_r = \delta_o.$$

From the equality of space in the turn left

$$S_{Oab} = S_{Oabc} - S_{Obc}$$

and

$$\frac{1}{2}\left(\sqrt{\left(R_1 - \frac{w}{2}\right)^2 + l^2}\right)\left(\sqrt{\left(R_1 + \frac{w}{2}\right)^2 + l^2}\right)\sin \vartheta = \frac{1}{2}\left(\left(R_1 + \frac{w}{2}\right) + a\right)l - \frac{1}{2}\left(R_1 + \frac{w}{2}\right)l$$

and

$$\left(\left(R_1^{\,2} - \left(\frac{w}{2}\right)^2\right)^2 + 2R_1^{\,2} l^2 + 2\left(\frac{w}{2}\right)^2 l^2 + l^4\right)\sin^2 \vartheta = (wl)^2$$

Further transformation leads to the following equation communication distance from the center of rotation O to the mid-point of the rear axle of the car with a control parameter ϑ

$$R_1 = \sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl \cot \vartheta}$$

Distance from the center of rotation O to the points a, b, c, d, C are defined by the equalities

$$R_a = \sqrt{\left(\sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl \cot \vartheta} - \frac{w}{2}\right)^2 + l^2}, \quad R_b = \sqrt{\left(\sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl \cot \vartheta} + \frac{w}{2}\right)^2 + l^2},$$

$$R_c = \sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl \cot \vartheta} + \frac{w}{2}, \quad R_d = \sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl \cot \vartheta} - \frac{w}{2},$$

$$R_C = \sqrt{\left(\sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl \cot \vartheta}\right)^2 + a_2^2}.$$

The steer angles of the front wheels of the vehicle δ_i and δ_o are connected with the radii curvature of the trajectory of its characteristic points and the control parameter by the following equations:

$$\delta_i = \cot^{-1}\frac{R_c}{l} = \cot^{-1}\left(\frac{R_1}{l} + \frac{w}{2l}\right),$$

$$\delta_o = \cot^{-1}\frac{R_d}{l} = \cot^{-1}\left(\frac{R_1}{l} - \frac{w}{2l}\right),$$

$$\delta_i = \cot^{-1}\frac{R_c}{l} = \cot^{-1}\left(\sqrt{\frac{w}{l}u + \left(\frac{w}{2l}\right)^2 - 1} + \frac{w}{2l}\right),$$

$$\delta_o = \cot^{-1}\frac{R_d}{l} = \cot^{-1}\left(\sqrt{\frac{w}{l}u + \left(\frac{w}{2l}\right)^2 - 1} - \frac{w}{2l}\right), \quad u = \cot\vartheta, \quad 0 < \vartheta \le \vartheta_{\max}.$$

The maximum Ackerman angle $\vartheta_{\max}$ is evaluated by the construction specialities of the vehicle and is connected to the maximum turning angle of the wheel $\delta_{\max}$ (43-45 degrees, in reality 40-41 degrees)

$$\vartheta_{\max} = \cot^{-1}\left(\frac{l}{w}\left[\left(\cot\delta_{\max} - \frac{w}{2l}\right)^2 + 1 - \left(\frac{w}{2l}\right)^2\right]\right).$$

4 Calculating the trajectory

Control law establishes analytical connection between a control parameter and the movement of any fixed point of the vehicle. Let us introduce into consideration the radius of curvature of the trajectory of motion of the vehicle (point C_1) by the following equality

$$\rho = l\sqrt{\frac{1}{4}\left(\frac{w}{l}\right)^2 + \frac{w}{l}\cot\vartheta - 1}, \ 0 < \vartheta \le \vartheta_{\max}. \tag{1}$$

As is known, any flat curve is completely determined by its natural equation [13]

$$k = k(s),$$

where s – a natural parameter (in this case, the distance travelled by car) $k = \frac{1}{\rho}$ – the curvature of the trajectory.кривизна траектории.

In the aims of formalizing the problem of the control of the vehicle we will determine the curvature of the trajectory of the vehicle by an equation

$$k = \frac{d\psi}{ds},$$

where the angle ψ – angle of curvature (the turning angle of the tangent to the trajectory of the movement of the vehicle), as the positive direction of the reference of the angle the left turning of the vehicle is chosen. When turning left $k(s) > 0, \rho(s) > 0$, when turning right $k(s) < 0, \rho(s) < 0$.

In this case the law of the control of a vehicle is expressed by the following equation:

$$\rho = \operatorname{sgn}\vartheta \, l\sqrt{\frac{1}{4}\left(\frac{w}{l}\right)^2 + \frac{w}{l}\cot|\vartheta| - 1}, \ -\vartheta_{\max} \le \vartheta \le \vartheta_{\max}. \tag{2}$$

Here $\vartheta = \delta_l - \delta_r$, when it's $-\vartheta_{\max} \le \vartheta < 0$ – the vehicle makes a right turn, but when it's $0 < \vartheta \le \vartheta_{\max}$ – a left turn.

I.e. any control functions $\vartheta(t)$ corresponds to only a trajectory of movement of the vehicle specified by equation (1). Conversely, any smooth trajectory of motion of the vehicle complies with the only function of the control, which is determined by the equation

$$\vartheta = \operatorname{sgn} k(s) \cot^{-1}\left(\frac{l}{w} + \frac{\rho^2}{wl} - \frac{w}{4l}\right) \tag{3}$$

Parametric equations of a trajectory in this case can be obtained in the following form:

$$x = \int_0^s \cos\left(\int_0^\sigma k(s)ds\right)d\sigma,$$

$$y = \int_0^s \sin\left(\int_0^\sigma k(s)ds\right)d\sigma.$$

When the vehicle is moving with a constant speed V , we obtain the following trajectories equations in the parametric form

$$x = \int_0^t V\cos\left(\int_0^\tau \frac{Vdt}{\operatorname{sign}\vartheta\, l\sqrt{\frac{1}{4}\left(\frac{w}{l}\right)^2 + \frac{w}{l}\cot\left|\vartheta(t)\right| - 1}}\right)d\tau,$$

$$y = \int_0^t V\sin\left(\int_0^\tau \frac{Vdt}{\operatorname{sign}\vartheta\, l\sqrt{\frac{1}{4}\left(\frac{w}{l}\right)^2 + \frac{w}{l}\cot\left|\vartheta(t)\right| - 1}}\right)d\tau.$$

5 Examples

For example, let us consider the dependence of the curvature radius of the trajectory of the vehicle ρ from the control parameter ϑ and assign the required space for the car to be able to move without touching the walls in a circular motion of the car (Fig. 4).

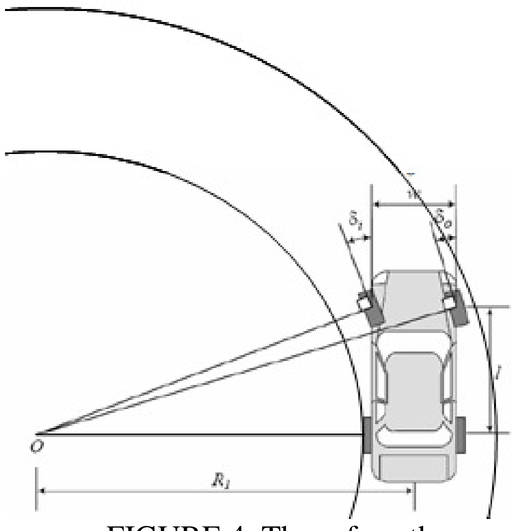

FIGURE 4. The safe pathway

The geometric parameters of some car models are given in Table 1.

Table 1. The geometric parameters of some car models [14]

Car	Vehicle length L (feet)	Wheelbase l (feet)	Overhang car a (feet)	Track w (feet)	w/l
2010 Honda Civic Coupe	14.625	8.692	2.566	5.742	0.660
2010 Chevy Impala	16.7	9.208	2.62	6.075	0.660
2009 Dodge Avenger	15.908	9.075	3.182	5.992	0.660
2010 Ford Taurus	16.908	9.408	2.114	6.35	0.67
2008 Chrysler Town & Country	16.875	10.1	2.813	6.408	0.63
2010 Ford Escape + Hybrid	14.642	8.592	2.486	5.925	0.69
2010 Honda Pilot	15.908	9.1	2.784	6.541	0.72
2010 Toyota Highlander Hybrid	15.7	9.15	2.962	6.267	0.68
2009 Saturn Vue Hybrid	15.008	8.883	2.832	6.067	0.68
Hummer H3	15.625	9.325	2.561	7.083	0.76
2009 Dodge Grand Caravan	16.875	10.1	2.837	6.408	0.63
2010 Ford Mustang	15.675	8.925	2.639	6.158	0.69
2010 Chevrolet Corvette	14.55	8.808	2.555	6.05	0.69
2010 Chevrolet Camaro	15.867	9.358	2.885	6.292	0.67
2010 Mercedes-Benz Sedan	17.208	10.383	2.581	6.958	0.67
2010 Lexus LS Sedan	16.992	10.142	2.604	6.15	0.61
2010 BMW Coupe	14.308	8.725	2.467	6.342	0.73
2010 BMW Sedan	14.85	8.892	2.5	6.608	0.74
2010 Ferrari California	14.967	8.758	2.686	6.242	0.71
2010 Volkswagen Beetle	13.425	8.233	2.573	5.658	0.69
2007 Mini Cooper	12.088	8.091	2.169	5.502	0.68
2010 Honda Accord Sedan	16.175	9.183	2.863	6.058	0.66
2010 Honda Odyssey	16.842	9.842	3.088	6.425	0.65
2009 Smart Car	8.842	6.125	1.215	5.115	0.86

The radii dependencies of curvature of the control parameter, obtained by the formula (1) are below (Fig. 5).

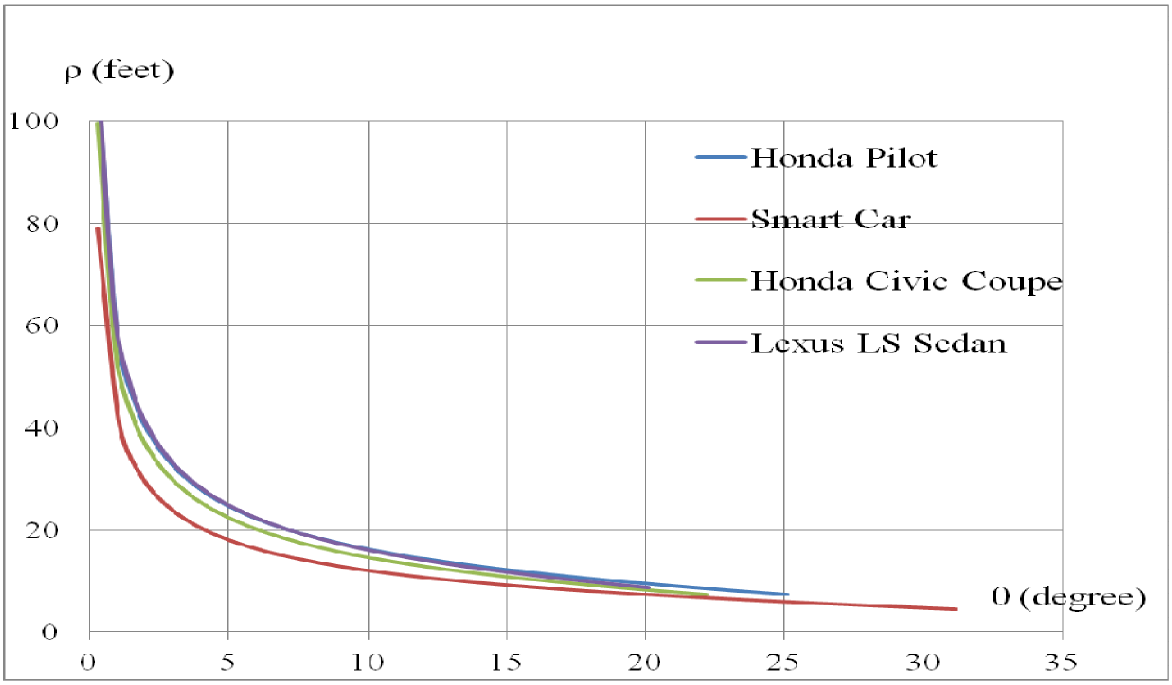

FIGURE 5. Dependence of radii of curvature of the control parameter for some cars. Максимальное значение угла Аккермана для всех марок автомобилей находилось из условия, что $\delta_{\max} = 40^{\circ}$.

The width of the safe pathway when the vehicle is moving is determined by the difference between the distances to the far point of the vehicle and to the middle of the wheel from the center of the rotation O (Fig. 4).

$$\Delta R = R_{\max} - R_{\min} \approx R_b - R_d = \sqrt{\left(\rho + \frac{w}{2}\right)^2 + (l+a)^2} - \left(\rho - \frac{w}{2}\right), \quad \rho(\vartheta_{\max}) \le \rho < \infty$$

and

$$\Delta R \approx \sqrt{\left(\sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl\cot\vartheta} + \frac{w}{2}\right)^2 + (l+a)^2} - \left(\sqrt{\left(\frac{w}{2}\right)^2 - l^2 + wl\cot\vartheta} - \frac{w}{2}\right), \quad 0 < \vartheta \le \vartheta_{\max} \qquad (4)$$

The dependence of the width of the safe pathway are shown on the control parameter, obtained by the formula (4) are shown in Figure 6.

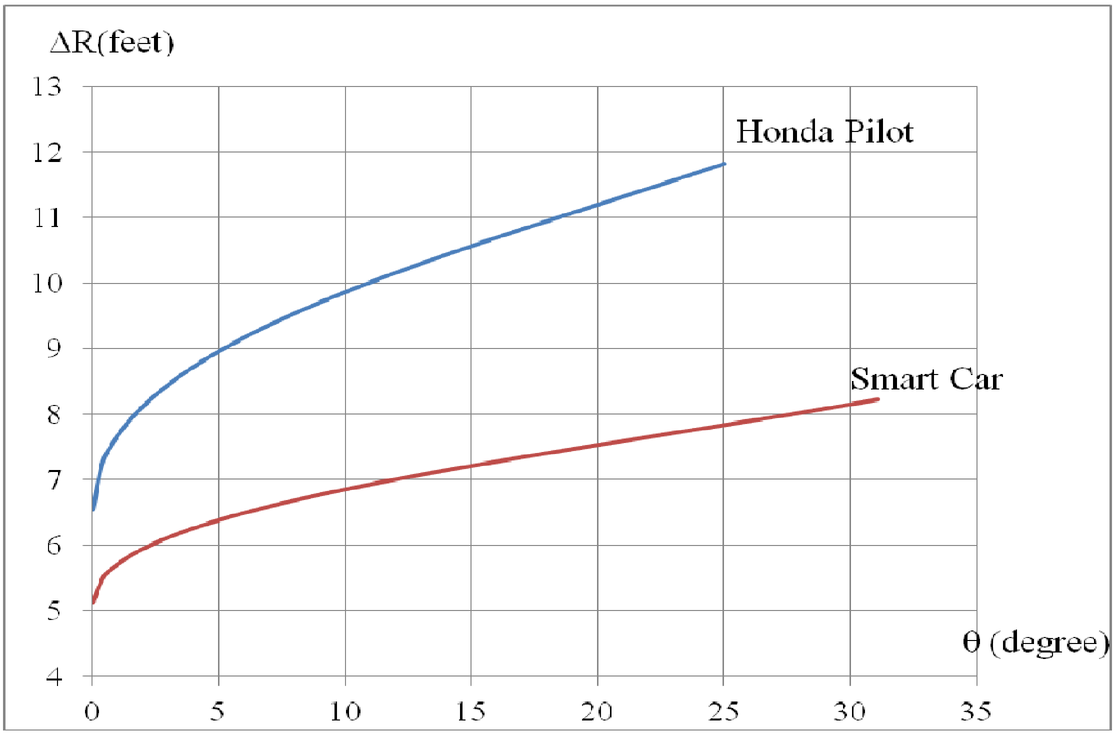

FIGURE 6. Width of the safe pathway for various values of the control parameter ϑ .

From Figures 5 and 6 the wall-to-wall turning circle D can be determined by the following equation:

$$D = 2\left(\rho + \frac{\Delta R}{2}\right).$$

For Smart Car and the Honda Pilot the turning circle is roughly equal to 18.2 and 25.8 feet respectively.

6 Conclusion

For the simplest model of a vehicle, when it is modeled as a rectangle, to obtain a universal one-parameter law of control of the vehicle (2), which allows to solve various kinematic and dynamic problems of its movement. Introduction of additional refining geometric characteristics of the vehicle does not make fundamental changes in the proposed mathematical model and may be implemented when necessary. The uses of the proposed model can be a significant refinement of algorithms of parallel parking, proposed in works [13-16], as well as the solution of navigation problems of management of motor vehicles using navigation systems GPS and GLONASS and problems of control of mobile robots with the help of tracking sensors. Formulas (1)-(4) may be useful in the design of motor roads, road interchanges, single-level and multilevel Parking lots, gasoline station, on-the-go fast food stations and the creation of car-simulators.

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